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COMMENT

Complexification and elliptic solution of the $A_1^{(1)}$ Toda chain parametrised by two arbitrary functions

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Abstract. The solutions of the $A_1^{(1)}$ Toda chain parametrised by two arbitrary functions are obtained with the use of a formal method of complexification from the earlier constructed real solutions.

One of the problems appearing in the theory of nonlinear differential equations is the problem of obtaining the most general solutions of corresponding equations in explicit form. As is well known, the general solution must be parametrised by the $2n$ arbitrary functions in the case of the n equations. In accordance with the ideology proposed by one of the authors (Zeitlin 1983, Chelnokov and Zeitlin 1983, 1984) it is reasonable to reject a number of arbitrary functions in the resulting expression and due to it to obtain a sufficiently simple formula for the solution. It might be of use for the applications. In this comment we demonstrate that the solution for the Liouville equation parametrised by two arbitrary functions and for the $A_1^{(1)}$ Toda chain also parametrised by two arbitrary functions can be obtained with the use of the solution from (Zeitlin 1983 and Chelnokov and Zeitlin 1983, 1984) introducing the formal method of complexification. The Liouville formula is naturally well known and only the method of obtaining it is of interest which, as can be seen from the example of the Toda chain given here, is generalised for the 'higher' Liouville equations. For the case of the Toda chain we obtain only half of the general solution. It should be noted that the formula for the general solution (Leznov and Saveliev 1980) is expressed in the form of an infinite series of very complicated structure and has limited applications.

The Lagrangian density of the one-dimensional chiral Kähler model (we shall consider two particular cases: O(3) and O(2, 1)σ-models) is:

$$L = h(u, \bar{u})(u_z\bar{u}_{\bar{z}} + u_{\bar{z}}\bar{u}_z), \tag{1}$$

where u is the initial complex chiral field ($\bar{u}$ is complex conjugate of u), parametrised by the points of the two-dimensional manifold $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, in the Euclidean case, and $z = x + y$, $\bar{z} = x - y$ in the pseudoeuclidean case. In the O(3) case $u = (n_1 + \mathrm{i}n_2)/(1 + n_3)$, where $n_1^2 + n_2^2 + n_3^2 = 1$, and in the O(2, 1) case $u = (1 - \mathrm{i}n_3)/(n_1 + n_2)$ where $-n_1^2 + n_2^2 + n_3^2 = -1$. $h = h(u, \bar{u})$ is the metric on the manifold,

$$h(u, \bar{u}) = (1 + u\bar{u})^{-2} \tag{2}$$

in the case of the O(3) σ-model, and

$$h(u, \bar{u}) = (u + \bar{u})^{-2} \tag{3}$$

in the case of the O(2, 1) σ-model. As was shown by one of us in Zeitlin (1981) and Bytsenko and Zeitlin (1982a, b) the reduction from the chiral model (1) to the new systems with the potential energy can be made and moreover the instanton sector or the anti-instanton sector of the initial model

$$u=u(z), \qquad u_{\bar z}=0 \qquad \text{or} \quad v=v(\bar z), \qquad v_z=0 \tag{4}$$

(u represents holomorphic and v anti-holomorphic functions) is reduced to the Liouville equation. That is, if the new dynamic variable $B(\tilde B)$ is defined by the formula

$$\exp B = hu_z\bar u_{\bar z}, \qquad \exp \tilde B = hv_{\bar z}\bar v_z \tag{5}$$

then the field B ($\tilde B$) satisfies the Liouville equation:

$$B_{z\bar z} \pm 2\exp B = 0 \tag{6}$$

where the upper sign corresponds to the O(3) σ-model, the lower to the O(2, 1) σ-model (and analogously for $\tilde B$ for the anti-instanton sector) and moreover the formulae (5), (2), (3) give the solution of equation (6) parametrised by the arbitrary (anti) holomorphic function (4). The solution of the equation of motion

$$hu_{z\bar z} + (\partial h/\partial u)u_z u_{\bar z} = 0 \tag{7}$$

corresponding to the Lagrangian density (1) and lying out of the instanton sector, generates the $A_1^{(1)}$ Toda chain (Lepowsky and Wilson 1978, Leznov and Saveliev 1980). If B^1, B^2 are the new dynamic variables

$$\exp B^1 = hu_z\bar u_{\bar z}, \qquad \exp B^2 = hu_{\bar z}\bar u_z, \tag{8}$$

then they satisfy the equations of the $A_1^{(1)}$ Toda chain (Zeitlin 1981, Bytsenko and Zeitlin 1982a, b)

$$\begin{aligned} &B^1_{z\bar z} \pm 2\exp B^1 \mp 2\exp B^2 = 0 \\ &B^2_{z\bar z} \mp 2\exp B^1 \pm 2\exp B^2 = 0. \end{aligned} \tag{9}$$

(The upper sign corresponds to the O(3) σ-model, the lower to the O(2, 1) σ-model). The formulae (8), (2), (3) give the solution of equations (9) if u is the solution of equation (7). It should be noted that the solutions of equation (7) for the O(3) and O(2, 1) σ-models in terms of the Jacobi elliptic functions and parametrised by arbitrary functions were constructed by one of us (Zeitlin 1983 and Chelnokov and Zeitlin 1983, 1984). They generalise the elliptic solution of Ghika and Visinescu (1982) and Abbott (1982).

We shall first consider the formal method of the complexification of the real solution for the example of the widely known Liouville equation. In accordance with (2), (3) and (5) we have for the solutions two formulae parametrised by arbitrary function ($u=u(z)$, $u_{\bar z}=0$):

$$\begin{aligned} &\exp B = u_z\bar u_{\bar z}/(1+u\bar u)^2 \qquad \text{(from O(3))}, \\ &\exp B = u_z\bar u_{\bar z}/(u+\bar u)^2 \qquad \text{(from O(2, 1))}. \end{aligned}$$

They are pure real solutions in accordance with (5). Let us make the formal substitution in (10)

$$u \to u^1(z), \qquad \bar u \to u^2(\bar z)$$

where now $\bar{u}^1 \neq u^2$. Then we have

$$\exp B = u^1_z u^2_{\bar{z}}/(1+u^1u^2)^2, \qquad \exp B = u^1_z u^2_{\bar{z}}/(u^1+u^2)^2. \tag{11}$$

It is obvious that formulae (11) satisfy the Liouville equation (6). Formulae (11) provide the general solution parametrised by two arbitrary functions $u^1 = u^1(z)$, $u^2 = u^2(\bar{z})$.

Now we consider the case of the $A^{(1)}_1$ Toda chain (9). In Zeitlin (1983) and Chelnokov and Zeitlin (1983, 1984) the following formulae for solutions (9) were obtained:

$$\begin{aligned} &\exp B^1 = \tfrac{1}{4}(kcn\phi + dn\phi)^2\phi_z\phi_{\bar{z}}, \\ &\exp B^2 = \tfrac{1}{4}(kcn\phi - dn\phi)^2\phi_z\phi_{\bar{z}}, \\ &dn\phi = dn(\phi, k), \qquad cn\phi = cn(\phi, k), \qquad 0<k\leqslant 1 \end{aligned} \tag{12}$$

(the upper sign in (9))

$$\begin{aligned} &\exp B^1 = \frac{1}{4}\frac{[c_1 + dn(\phi/k)/k]^2}{cn^2(\phi/k)}\phi_z\phi_{\bar{z}}, \\ &\exp B^2 = \frac{1}{4}\frac{[c_1 - dn(\phi/k)/k]^2}{cn^2(\phi/k)}\phi_z\phi_{\bar{z}}, \\ &k = (1+c_1^2)^{-1/2}, \qquad 0<k\leqslant 1, \end{aligned} \tag{13}$$

(the lower sign in (9)), where

$$\phi = \phi(z, \bar{z}), \qquad \phi = \bar{\phi}, \qquad \phi_{z\bar{z}} = 0. \tag{14}$$

For example,

$$\phi = \tfrac{1}{2}\ln f\bar{f}$$

where

$$f = f(z), \qquad f_{\bar{z}} = 0 \tag{15}$$

or

$$\phi = (1/2\mathrm{i})\ln(g/\bar{g})$$

where

$$g = g(z), \qquad g_{\bar{z}} = 0.$$

Let us make the formal complexification. In (14) let $\phi \neq \bar{\phi}$, that is ϕ is the arbitrary harmonic function (it is not necessary that it should be real):

$$\phi = \phi(z, \bar{z}), \qquad \phi_{z\bar{z}} = 0, \qquad \phi \neq \bar{\phi}. \tag{16}$$

It is easy to see that (12), (13) and (16) give the solutions of equations (9). Instead of formulae (15) we have for example:

$$\begin{aligned} &f \to f_1(z), \qquad \bar{f} \to f_2(\bar{z}), \qquad g \to g_1(z), \qquad \bar{g} \to g_2(\bar{z}) \\ &\phi = \tfrac{1}{2}\ln f_1 f_2, \qquad f_1 = f_1(z), \qquad f_2 = f_2(\bar{z}), \qquad f_1 \neq \bar{f}_2 \end{aligned} \tag{17}$$

or

$$\phi = (1/2\mathrm{i})\ln(g_1/g_2), \qquad g_1 = g_1(z), \qquad g_2 = g_2(\bar{z}), \qquad g_1 \neq \bar{g}_2$$

analogously to (15) or in general

$$\phi = \phi_1(z) + \phi_2(\bar{z}), \qquad \phi_1 = \phi_1(z), \qquad \phi_2 = \phi_2(\bar{z})$$

That is formulae (12), (13), (16) and (17) *give the solutions of equations (9) parametrised by two arbitrary complex functions.* Now these solutions are already complex. The condition $B^1 = -B^2 = B$ gives the formulae for the solution of the sinh-Gordon equation analogous to those given in Zeitlin (1983) and Chelnokov and Zeitlin (1983, 1984).

It should be noted that in the *particular* case of this construction, when we consider the one-dimensional chains (9), that is if in the formulae (9), (12) and (13) we have $z = \bar{z} = x$, $\phi = \bar{\phi}$ then from (14) we get $\phi_{xx} = 0$ and $\phi = ax + b$ and (12) and (13) provide the solutions for the one-dimensional particular case of system (9).

In a separate publication, one of us (MGZ) gives the method of construction of the infinite sequence of analogous formulae in terms of elliptic functions and parametrised by two arbitrary functions.

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