GENERALIZED POHLMEYER TRANSFORMATION – SOME NONLINEAR SYSTEMS WITH EXPONENTIAL INTERACTIONS

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The Pohlmeyer transformation between the O(3) nonlinear σ-model and the sine-Gordon equation is generalized to the case of the chiral model on a Kähler manifold. As a result nonlinear systems with the exponential interactions are obtained. Solutions of these systems are derived from solutions of the initial chiral model.

The correspondence between solutions of the O(3) σ-model and the sine-Gordon equation was established by Pohlmeyer [1]. In this work we will generalize Pohlmeyer's results to an arbitrary two-dimensional euclidean Kähler chiral model. We establish a local equivalence between some nonlinear systems with exponential interaction and systems without potential energy, defined on a manifold with a non-euclidean metric. This equivalence allows us to get the solution of the final model from that of the initial model. We consider the transformation from the Euler equations (for which in the CP^n case solutions with finite action [2] and meron solutions [3] were recently obtained), as well as from the instanton sector of the initial chiral model to the final equations.

1. Generalized Pohlmeyer transformation. The action A of the euclidean Kähler chiral model can be put in the form (we use the notations from ref. [4]):

$$A = \sum_{i,j=1}^{n} \int h^{ij}(u^j_z \bar{u}^i_{\bar{z}} + u^j_{\bar{z}} \bar{u}^i_z)\,\mathrm{d}^2x .$$

Let us denote

$$A^{ij}_k = h^{ij}\psi^{ij}_k, \quad k = 1,2, \tag{1}$$

$$\psi^{ij}_1 = \bar{u}^i_{\bar{z}} u^j_z, \quad \psi^{ij}_2 = \bar{u}^i_z u^j_{\bar{z}} \tag{2}$$

(sum over repeated indices in the case of the symbol Σ only). In Pohlmeyer's paper the lagrangian of initial model was considered as a new dynamic variable. Unlike Pohlmeyer we take the components A^{ij}_k of the lagrangian as the new variables. Deriving the equation for A^{ij}_k, we use the equation of motion for the chiral fields u^i, $1 \leqslant i \leqslant n$:

$$\sum_i h^{ij} u^i_{z\bar{z}} + \sum_i \sum_k h^{ij}_{,u^k} u^i_{\bar{z}} u^k_z = 0. \tag{3}$$

It is easy to obtain the expression for $A^{ij}_{1,z\bar{z}}$ from definition (1)

$$\begin{aligned}(\ln A^{ij}_1)_{z\bar{z}} &= (\ln h^{ij})_{z\bar{z}} \\ &+ (h^{ij}/A^{ij}_1)(\bar{u}^i_{z\bar{z}\bar{z}} u^j_z + u^j_{zz\bar{z}} \bar{u}^i_{\bar{z}} + 2\bar{u}^i_{z\bar{z}} u^j_{z\bar{z}}) \\ &+ (h^{ij}/A^{ij}_1)_z u^j_{z\bar{z}} \bar{u}^i_{\bar{z}} + (h^{ij}/A^{ij}_1)_{\bar{z}} \bar{u}^i_{z\bar{z}} u^j_z .\end{aligned} \tag{4}$$

Using the substitution $1 \rightleftharpoons 2$, $u^i \rightleftharpoons \bar{u}^j$ we can obtain from eq. (4) the expression for A^{ij}_2. For further transformation of the right-hand side of eq. (4) it is necessary to use the equations of motion (3), their differentiation and the particular form of metric h^{ij}. As a result the right-hand side of eq. (4) contains only functions of A^{ij}_k.

2. The equations generated by the instanton sector. (i) At first let us consider eq. (4) in the (anti-) instanton sector: $u^i_{\bar{z}} = 0$ ($v^i_z = 0$). The duality equations have nontrivial solutions when a nontrivial holomorphic mapping of CP^1 into the Kähler manifold exists. This complies with the discussion of the simply connected compact homogeneous Kähler manifolds. Using the duality equations, from eq. (4) we obtain:

$$B^{ij}_{z\bar{z}} = (\ln h^{ij})_{z\bar{z}}, \tag{5}$$

where $\exp B^{ij} = A^{ij} \equiv A^{ij}_1$. Making use of antiduality equations we can get:

$$\widetilde{B}^{ij}_{z\bar{z}} = (\ln h^{ij})_{z\bar{z}}, \quad \exp \widetilde{B}^{ij} = \widetilde{A}^{ij} \equiv A^{ij}_2.$$

At last we obtain an expression for $(\ln h^{ij})_{z\bar{z}}$ in terms of B^{ij} so that one can get the matrix equation and its solutions, which are easy to obtain from instanton solutions of the initial model:

$$\exp B^{ij} = h^{ij}(u^i, \bar{u}^j) u^j_z \bar{u}^i_{\bar{z}}, \quad u^i_{\bar{z}} = 0,$$

$$\exp \widetilde{B}^{ij} = h^{ij}(v^i, \bar{v}^j) \bar{v}^i_z v^j_{\bar{z}}, \quad v^i_z = 0. \tag{6}$$

(ii) Let us consider the CP^1 model with metric $h = (1 + u\bar{u})^{-2}$ as an example. Then the expression (5) is just the Liouville equation

$$B_{z\bar{z}} + 2 \exp B = 0. \tag{7}$$

Thus eqs. (6) with $i = j = 1$ give its solution (parametrized by holomorphic u and antiholomorphic v functions), which is just the Liouville formula for real solutions of the Liouville equation.

(iii) Eq. (5) with $i = j = 1$ and dual equations $u_{\bar{z}} = 0$ gives not only the Liouville equation, but also e.g., the Dodd–Bullough equation. All the one-dimensional Kähler metrics are generated by the potential P: $h = \partial^2 \ln P/\partial u\, \partial \bar{u}$. Let $P = P(u\bar{u}) = P(y), y = u\bar{u}$ be an arbitrary Kähler potential and D be the following operator

$$DP = (P_{yy}P - P_y^2)y + P_y P. \tag{8}$$

Then the expression (5) can be rewritten in the form

$$B_{z\bar{z}} + 2 \exp B = D^2 P (DP)^{-2} y_{z\bar{z}}. \tag{9}$$

One can construct from eq. (9) any nonlinear equation with the exponential interaction and expression (6) gives a solution of this equation. The function u, parametrizing this solution, is not an arbitrary holomorphic function, but satisfies some constraints. Let P be equal to $1 + y + y^2$, then it follows from eq. (9) that

$$B_{z\bar{z}} + 2 \exp B - 4\,\Phi \exp(-2B) = 0, \tag{10}$$

where $\Phi = (1 + y + y^2)^{-3} y^3_{z\bar{z}}$. The constraint $\Phi = 1$ gives the Dodd–Bullough equation with the solution

$$\exp B = DP \cdot P^{-2} y_{z\bar{z}} = h u_z \bar{u}_{\bar{z}}, \tag{11}$$

where u is an arbitrary holomorphic solution of $u_z \bar{u}_{\bar{z}} = 1 + u\bar{u} + u\bar{u}^2$. The metric takes form $h = DP \cdot P^{-2} = (1 + 4y + y^2)(1 + y + y^2)^{-2}$. Moreover, for any polynomial

$$P = 1 + \sum_{i=1}^{n} y^i$$

with degree $n \geqslant 2$ one can get eq. (10) and the solution (11), but the constraint Φ for $n \geqslant 3$ gives a transcendental equation for u. Note that for $P = 1 + y$ one obtains Liouville equation and for $P = \exp y$ the Laplace equation. One can give a geometrical meaning to eq. (5) ($n = 1$). Its right-hand side is equal to

$$(\ln h)_{z\bar{z}} = (\partial^2 \ln h/\partial u\, \partial \bar{u}) u_z \bar{u}_{\bar{z}}$$

(where u is a holomorphic mapping $CP^1 \to M^1$), which coincides with the Ricci form density of the hermitian manifold M^1 [5]:

$$\mathrm{Ric}\, \Omega = \tfrac{1}{2} i (\partial^2 \ln h/\partial u\, \partial \bar{u})\, du \wedge d\bar{u}.$$

(iv) Now we consider eqs. (5) and (6) in the instanton sector for the CP^n model with the Fubini–Study metric:

$$h^{jk}_{CP^n}(u^1, \ldots, u^n) = \left(1 + \sum_{i=1}^{n} u^i \bar{u}^i\right)^{-1} \times \left[\delta^{jk} - \left(1 + \sum_{i=1}^{n} u^i \bar{u}^i\right)^{-1} u^j \bar{u}^k\right]. \tag{12}$$

Using instead of (6) the solutions

$$\exp B^{ij}_n = a^{ij}_n h^{ij}_{CP^n}(u^1, \ldots, u^n) \bar{u}^i_{\bar{z}} u^j_z,$$

$$\exp \widetilde{B}^{ij}_n = a^{ij}_n h^{ij}_{CP^n}(v^1, \ldots, v^n) \bar{v}^i_z v^j_{\bar{z}}, \tag{13}$$

where $a^{ij}_n \in C\backslash 0$ and taking into account (12), the eqs. (5) can be rewritten as follows

$$B^{ij}_{n,z\bar{z}} + \sum_{i,j\in\{1,...,n\}} K^{ij}_n \exp B^{ij}_n$$

$$-\tfrac{1}{2}\delta^{ij} \sum_{r,s\in\{1,...,\hat{i},...,n\}} K^{rs}_{n-1} \exp B^{rs}_{n-1} = 0\,, \qquad (14)$$

where $K^{ij}_n = 2/a^{ij}_n$, $1 \leqslant i,j \leqslant n$, $1 \leqslant r,s \leqslant n-1$. The notation $r,s \in \{1, ..., \hat{i}, ..., n\}$ means that r and s take their values in $1, 2, ..., n$ with i omitted. The recursive eqs. (14) for n^2 fields B^{ij}_n contain an "inhomogeneous" part (the last term in the left-hand side of eq. (14)). The inhomogeneous part is constructed from the solutions of the system with the dimension less by one. Let us briefly describe the recurrence procedure. On the nth step the system (14) for fields B^{ij}_n have to be added by n system of equations for B^{rs}_{n-1} (each system contains $(n-1)^2$ equations).

The geometric reason is that n copies CP^{n-1} are contained in CP^n. If, instead of fields B^{ij}_n satisfying the recurrence eqs. (14), we consider a set of fields ρ^s in the case of $n = 2$, the system (14) takes the form [6]:

$$\rho^i_{z\bar{z}} + \sum_{s=1}^{6} P^{is} \exp \rho^s = 0\,, \qquad (15)$$

where

$$P^{is} = \begin{pmatrix} K^{11}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & K^{22}_1 & 0 & 0 & 0 & 0 \\ 0 & -\tfrac{1}{2}K^{22}_1 & K^{11}_2 & K^{12}_2 & K^{21}_2 & K^{22}_2 \\ 0 & 0 & K^{11}_2 & K^{12}_2 & K^{21}_2 & K^{22}_2 \\ 0 & 0 & K^{11}_2 & K^{12}_2 & K^{21}_2 & K^{22}_2 \\ -\tfrac{1}{2}K^{11}_1 & 0 & K^{11}_2 & K^{12}_2 & K^{21}_2 & K^{22}_2 \end{pmatrix} \qquad (16)$$

and $B^{11}_1 = \rho^1, B^{22}_1 = \rho^2, B^{11}_2 = \rho^3, B^{12}_2 = \rho^4, B^{21}_2 = \rho^5, B^{22}_2 = \rho^6$. It should be noted that matrix (16) is of non-Cartan form [6]. However, we can get the solutions with the formulae (13). These solutions are parametrized by arbitrary analytic functions (but it is not enough for integrability in accordance with [6]). However, the Lax representation exists for any numerical matrix $P^{\alpha\beta}$ and the formulae (13) analogous to formulae (6). Matrix systems similar in structure are generated by polydiscs and by spheres with the Bergman metric and by the unit disc without {0} with the Kobayashi metric.

3. Equations generated by equations of motion (3) of the CP^n model. (i) We consider the CP^1 model at first. The following system is obtained from eq. (4)

$$\tilde{B}^1_{z\bar{z}} + \alpha \exp \tilde{B}^1 - \beta \exp \tilde{B}^2 = 0\,,$$

$$\tilde{B}^2_{z\bar{z}} - \alpha \exp \tilde{B}^1 + \beta \exp \tilde{B}^2 = 0\,, \qquad (17)$$

where

$$\exp B^1 \equiv A^{11}_1\,, \quad \exp B^2 \equiv A^{11}_2\,,$$

$$\exp \tilde{B}^1 = 2\alpha^{-1} \exp B^1 = 2\alpha^{-1} h_{CP^1}(u) u_z \bar{u}_{\bar{z}}\,,$$

$$\exp \tilde{B}^2 = 2\beta^{-1} \exp B^2 = 2\beta^{-1} h_{CP^1}(u) u_{\bar{z}} \bar{u}_z\,. \qquad (18)$$

Here u is an arbitrary solution of the Euler equation of the CP^1 model. In the case $\alpha = \beta = 2$ one of the two-dimensional Cartan matrices of the Kac–Moody algebra [6] is obtained and for $\tilde{B}^2 = -\tilde{B}^1$ the system (17) is reduced to the sinh-Gordon equation. For $n = 1$ the solutions with finite action are instantons only [2]. Let us show how to construct a solution of this equation. There exists a conservation law for the CP^1 model $hu_z\bar{u}_z = F(z)$ (the case $F(z) = \exp i\alpha$, α = const. corresponds to the sinh-Gordon equation). If the topological charge does not vanish, we can solve the conservation law equation instead of the equation of motion. Then we substitute u into (18) and obtain the solution of (17) (in one-dimensional case it is easy to obtain the solution in terms of elliptic functions).

(ii) For $n \geqslant 2$ there exist solutions with finite action, which are not (anti-)instantons. In this case the transformation of eq. (4) can easily be performed only approximately in the following way. An exact derivation of this transformation encounters considerable difficulties as the fields u^i are mixed up in the first sum of the second term of eq. (3). Let u^i be a small solution of eq. (3). Then, recalling that

$$h^{ii} \propto 1 + \sum u^k \bar{u}^k\,, \quad k \in \{1, ..., \hat{i}, ..., n\}$$

and

$$h^{ij} \propto u^i \bar{u}^j \quad (i \neq j)\,,$$

we can neglect all terms containing h^{ij} with $i \neq j$. More rigorously, if $HB = 0$ is the reduced equation and $B = B(u^\mu, u^\nu, u_z^\mu, \bar{u}_{\bar{z}}^\nu)$ is its solution then the operator H has the structure

$$H = H_0 + H_1, \quad H_0 \propto O(1 + u^\mu u^\nu) u_z^\mu \bar{u}_{\bar{z}}^\nu,$$

$$H_1 \propto O(u^\mu u^\nu) u_z^\mu \bar{u}_{\bar{z}}^\nu .$$

The condition $|u^\mu u^\nu| \ll 1$ permits to consider the approximate form of the reduced equation: $H_0 B = 0$. After our transformation we have only $2n$ quantities on the diagonal ($i = j$) instead of $2n^2$ in eqs. (1). In a similar way one can obtain from eq. (4) with $k = 2$ the system with a matrix (16). However, the block standing on the place K_2 is constructed from the blocks M_1 and M_2

$$\begin{pmatrix} M_1 & M_2 \\ M_1 & M_2 \end{pmatrix}, \quad M_i = \begin{pmatrix} \alpha_i & -\beta_i \\ -\alpha_i & \beta_i \end{pmatrix}. \tag{19}$$

For arbitrary n one obtains n types of blocks. It follows from this structure that the system with non-Cartan matrix has exact solutions (1), but the number of these solutions is not sufficient for the complete integrability [6].

4. Conclusion. The proposed method can be used to generate from the chiral models on one dimensional Kähler manifolds all models of a given class [Liouville (7), Dodd-Bullough (10), sinh-Gordon (17)]. From multidimensional manifolds one can obtain the systems of non-Cartan type both in the instanton sector (exactly) (14)–(16) and from the complete equation of motion (approximately) (19). This method gives the information about solutions of the derived equations both explicitly and in an indirect way. The derivation was essentially local as the decomposition procedure of the lagrangian to the components A_k^{ij} is not invariant. All constructions were done for an arbitrary topological charge of the chiral field u^ν. The obtained systems possess the hereditary dual symmetry U^γ, which is inherent in nonlinear σ-models on symmetric spaces [7]. The transformations of the reduced systems which follow from U^γ are very important for noninstanton solutions as the instanton solutions are fixed points for U^γ. Besides, using formulae from ref. [8] for the finite action solutions of the $O(2k + 1)$ σ-model we can transform them into solutions of the corresponding reduced systems using generalization of the results of ref. [9]. These points will be considered in detail together with the reduction of supersymmetric chiral models in a separate publication. The correspondence between $O(n)$ and $U(n)$ models and nonlinear systems with potential energy was established also in terms of L–**A** pairs [10] and by the Lund–Regge method [11].

This paper represents the further development of ref. [9].

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