VORTEX SOLUTIONS IN THE MODEL OF ^{3}He FILMS AND THE O(3) σ-MODEL

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By using the reduction from the microscopic model of ^{3}He films to the chiral O(3) σ-model possible vortex solutions are considered.

1. Recently there have been experimental studies of thin ^{3}He films one or two atoms in thickness [1]. In ref. [2] a microscopic model of the ^{3}He film was suggested, based on the 3-dimensional model of ^{3}He developed earlier [3]. As shown in ref. [2], in two dimensions there may exist several superfluid phases, two of them being energetically preferable and stable under small fluctuations (the so-called a and b phases). The Bose-spectrum of the system in question contains both phonon and nonphonon branches. The order parameter in such a theory is a 2×3 matrix. The Bose-condensate cannot exist in two-dimensional systems at finite temperatures. Nevertheless we can expect such a system to be a superfluid because the long-range correlations decrease more slowly than exponentially at large distances. In ref. [2] it was demonstrated that if we introduce an analog of polar coordinates for tensor Bose-fields, we can show that the long-range behaviour of the field correlator is determined by the behaviour of the correlator of angle variables, as in the two-dimensional superfluidity theory [4]. The number of such angle variables is equal to the number of phonon modes of the system. The angle variables in the a-phase are the angle $\varphi \in S^1$ and the vector $\boldsymbol{n} \in S^2 \subset R^3$.

The action functional for "slow" fields is

$$S = -\int (a\, \partial_i \varphi\, \partial_i \varphi + b\, \partial_i n_a\, \partial_i n_a)\, d^2x . \tag{1}$$

Here a, b are real constants, $i = 1, 2, a = 1, 2, 3$.

So in this approach the problem of the long-range behaviour of correlators for ^{3}He-films is reduced to the investigation of the O(3) σ-model [5]. In particular, the existence or nonexistence problem of superfluidity is reduced to the problem of the long-range behaviour of the two-point correlator in the O(3) σ-model.

It would also be interesting to investigate classical solutions corresponding to the action (1), in particular, possible vortex solutions (periodic with respect to the coordinates of the fields n_a).

2. As one can see from (1), the field φ is free and unrelated to the n_a fields, so only the $\boldsymbol{n}$ field is of interest.

In order to build up vortex solutions in the model under consideration we shall follow ref. [6] where two Ansätze for solutions of the O(3) σ-model parametrized by an arbitrary function are considered. Let us introduce the complex field

$$u = (n_1 + in_2)/(1 + n_3) , \tag{2}$$

instead of the n_a-fields. The equation of motion corresponding to the action (1) is

$$hu_{z\bar{z}} + (\partial h/\partial u)\, u_z u_{\bar{z}} = 0 . \tag{3}$$

Here $h = (1 + u\bar{u})^{-2}$ is a metric on the sphere S^2, $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$, $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), x_1, x_2$ are coordinates on the euclidean plane R^2.

We shall apply the first Ansatz in accordance with ref. [6] in the form

$$u = A(x) \exp[iB(y)] , \tag{4}$$

where

$$x = x(z, \bar{z}), \quad y = y(z, \bar{z}), \quad x_{z\bar{z}} = y_{z\bar{z}} = 0 ,$$

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$$x_z y_{\bar z} + x_{\bar z} y_z = 0, \quad x_z x_{\bar z} = y_z y_{\bar z}, \quad x = \bar x, \quad y = \bar y . \tag{5}$$

We do not constrain the field u by any conditions, and after the system (4), (5) is solved we obtain a formula parametrized by an arbitrary holomorphic function. Considering only the simplest case (in the general case there are three constants more), we have

$$u = [(1 + k \,\mathrm{sn}\, x)/(1 - k \,\mathrm{sn}\, x)]^{1/2} \exp(\mathrm{i}y) .$$

The system (4), (5) is satisfied, for instance, if we set

$$x = \tfrac{1}{2} \ln f\bar f, \quad y = (1/2\mathrm{i}) \ln (f/\bar f) . \tag{6}$$

Here $f = f(z), f_{\bar z} = 0$ is an arbitrary holomorphic function, $k \in [0,1]$ is the parameter of the Jacobi elliptic function sn.

In the case $f \sim z^n$ this solution was considered in ref. [7], but the general solution was not obtained there because the field u was required to have a fixed asymptotic behaviour at $z \to \infty$. For instance, in ref. [7] it was impossible to consider the case $f \sim \exp z$ which we discuss here, and it was impossible to use the solution with a functional parameter.

So we obtain

$$u = \left(\frac{1 + k \,\mathrm{sn}(\frac{1}{2} \ln f\bar f)}{1 - k \,\mathrm{sn}(\frac{1}{2} \ln f\bar f)}\right)^{1/2} (f/\bar f)^{1/2} . \tag{7}$$

The second Ansatz is

$$u = A(x) \exp[\mathrm{i}B(x)] ,$$
$$x = x(z, \bar z), \quad x = \bar x, \quad x_{z\bar z} = 0 , \tag{8}$$

(in ref. [6] it was considered for the case of the O(2, 1) σ-model and the Ernst equation).

It also gives the solution, parametrized by an arbitrary function

$$u = \left(\frac{1 + c \sin x}{1 - c \sin x}\right)^{1/2} \exp[\mathrm{i} \arctan(s \tan x)]$$
$$= \frac{\cos x + \mathrm{i}s \sin x}{1 - c \sin x} . \tag{9}$$

Here c, s are real constants, $c^2 + s^2 = 1$, and x is given by (6).

3. The general solutions (7), (9) also contain periodic configurations (in the coordinates x_1, x_2). One can regard them as vortex solutions. For the $\boldsymbol{n}$ field solutions given by (7) in the special case $f = \exp(c_1 + \mathrm{i}c_2)z$ we have

$$n_1 = \mathrm{dn}(c_1 x_1 - c_2 x_2) \cos(c_1 x_2 + c_2 x_1) ,$$
$$n_2 = \mathrm{dn}(c_1 x_1 - c_2 x_2) \sin(c_1 x_2 + c_2 x_1) ,$$
$$n_3 = -k \,\mathrm{sn}(c_1 x_1 - c_2 x_2) . \tag{10}$$

Here c_1, c_2 are some constants, dn is the elliptic Jacobi function. If $c_2 = 0$ ($c_1 = 0$) we found the elliptic behaviour along one axis x_1 (x_2) and the trigonometric one along the other axis x_2 (x_1).

The solution (9) gives

$$n_1 = \cos x, \quad n_2 = s \sin x, \quad n_3 = -c \sin x,$$
$$c^2 + s^2 = 1 . \tag{11}$$

Taking the same function f we have $x = c_1 x_1 - c_2 x_2$ in (11).

Let us note that (10) depends on two variables, whereas (11) contains only one variable. The vortex lattices (10), (11) in some sense are the analogs of the Abrikosov lattice in the theory of superconductivity. The stable superfluid state may arise if the condition of the action extremum (1) is fulfilled. This is the case for the classical solutions (10), (11).

The absolute minimum of the action (1) is realized on the instanton solution corresponding to the harmonic mapping $S^2 \to S^2$. In our case we deal with the mapping $R^2 \to S^2$ and obtain the instanton solution from the general formula (7) if we take $k = 1$,

$$u = f(z) = \prod_{i=1}^{q} \frac{z - a_i}{z - b_i} .$$

We hope to discuss the stability problem for the solutions (10), (11), as well as the one for vortex solutions in the b-phase (where there exist two "connected" O(3) σ-models), in a separate paper.

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